Link Analysis

CSE545 - Spring 2022

Stony Brook University

H. Andrew Schwartz

Big Data Analytics, The Class

Goal: Generalizations
A *model* or *summarization* of the data.

Data Frameworks

- Hadoop File System ✓
- Spark ✓
- Streaming ✓
- MapReduce ✓
- Tensorflow ✓

Algorithms and Analyses

- Similarity Search ✓
- Hypothesis Testing
- **Link Analysis**
- Recommendation Systems
- Deep Learning

The Web, circa 1998

The Web, circa 1998

Match keywords, language (*information retrieval*)

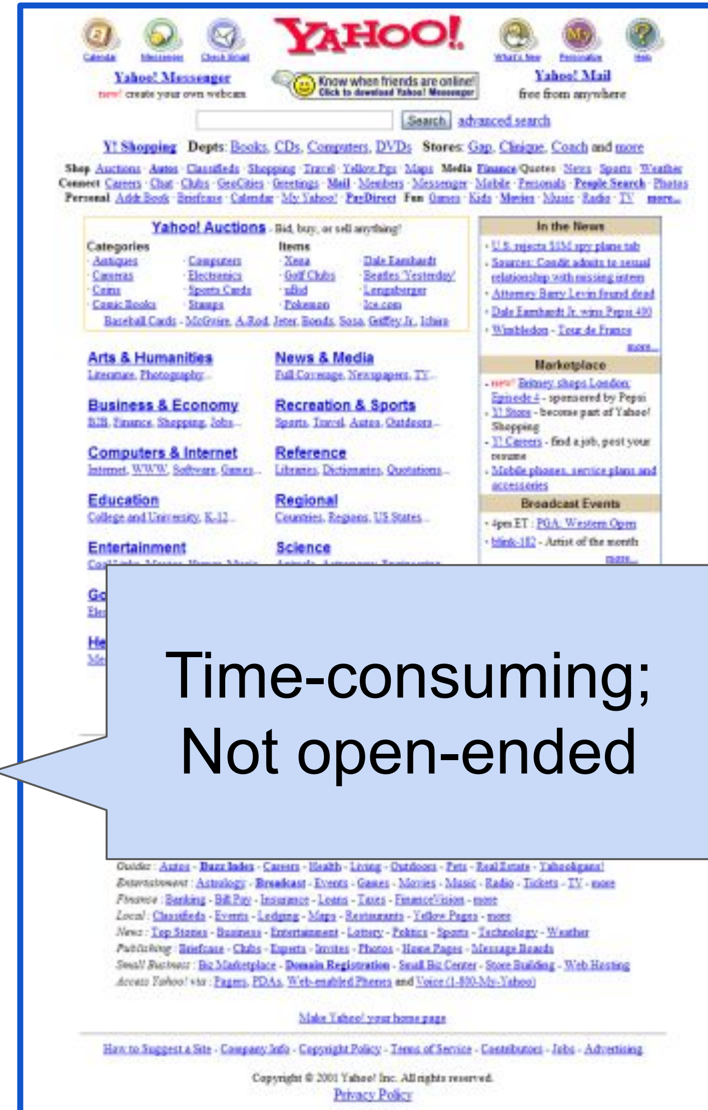
Explore directory

The Web, circa 1998

Easy to game with "term spam"

Match keywords, language (*information retrieval*)

Explore directory



Time-consuming; Not open-ended

Enter PageRank

The Anatomy of a Large-Scale Hypertextual Web Search Engine

Sergey Brin and Lawrence Page

Computer Science Department,
Stanford University, Stanford, CA 94305, USA
sergey@cs.stanford.edu and page@cs.stanford.edu

Abstract

In this paper, we present Google, a prototype of a large-scale search engine which makes heavy use of the structure present in hypertext. Google is designed to crawl and index the Web efficiently and produce much
text and hyperlink

...

The PageRank Citation Ranking: Bringing Order to the Web

January 29, 1998

Abstract

PageRank

Key Idea: Consider the citations of the website.

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Who links to it?
and what are
their citations?

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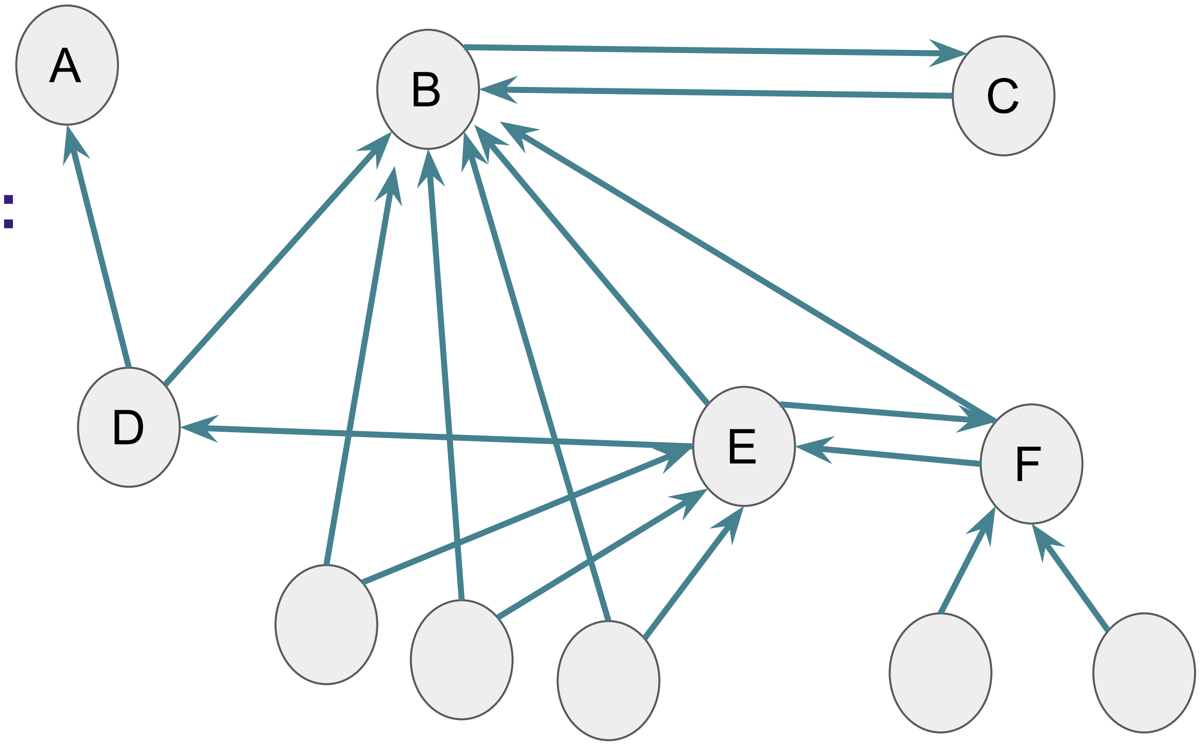
Innovation 1: What pages would a “random Web surfer” end up at?

Innovation 2: Not just own terms but what terms are used by citations?

PageRank

View 1: Flow Model:
in-links as votes



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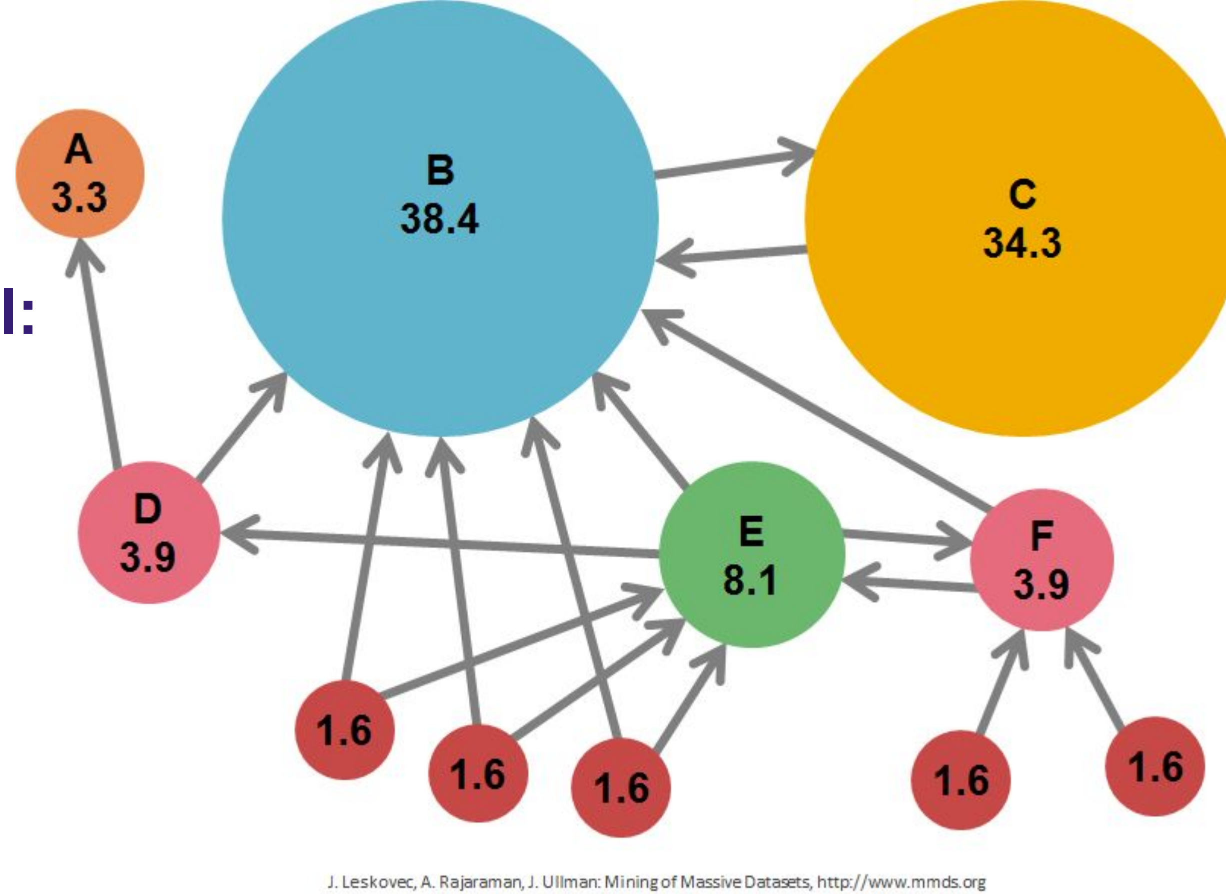

View 1: Flow Model:
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Innovation 1: What pages would a "random Web surfer" end up at?

Innovation 2: Not just own terms but what terms are used by citations?

PageRank

View 1: Flow Model:

in-links (citations) as votes

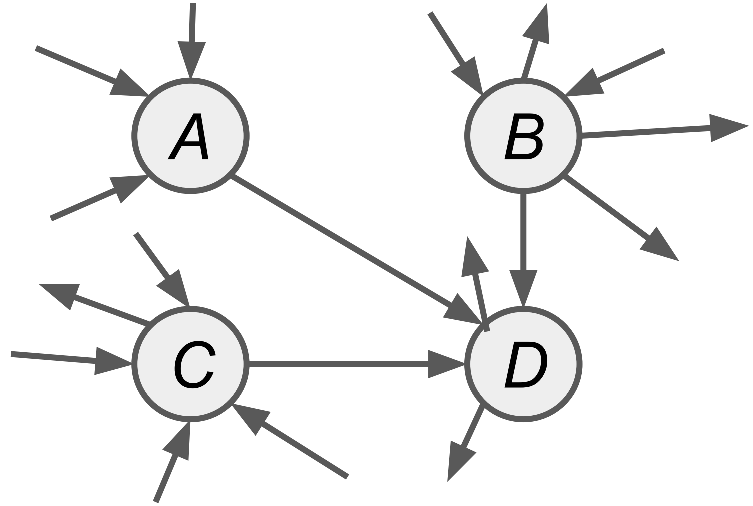
but, citations from important pages should count more.

=> Use recursion to figure out if each page is important.

Innovation 1: What pages would a “random Web surfer” end up at?

Innovation 2: Not just own terms but what terms are used by citations?

PageRank



View 1: Flow Model:

How to compute?

Each page (j) has an importance (i.e. rank, r_j)

$$vote_j = \frac{r_j}{n_j}$$

(n_j is |out-links|)

$$r_j = \sum_{i \in inLinks(j)} vote_i$$

PageRank

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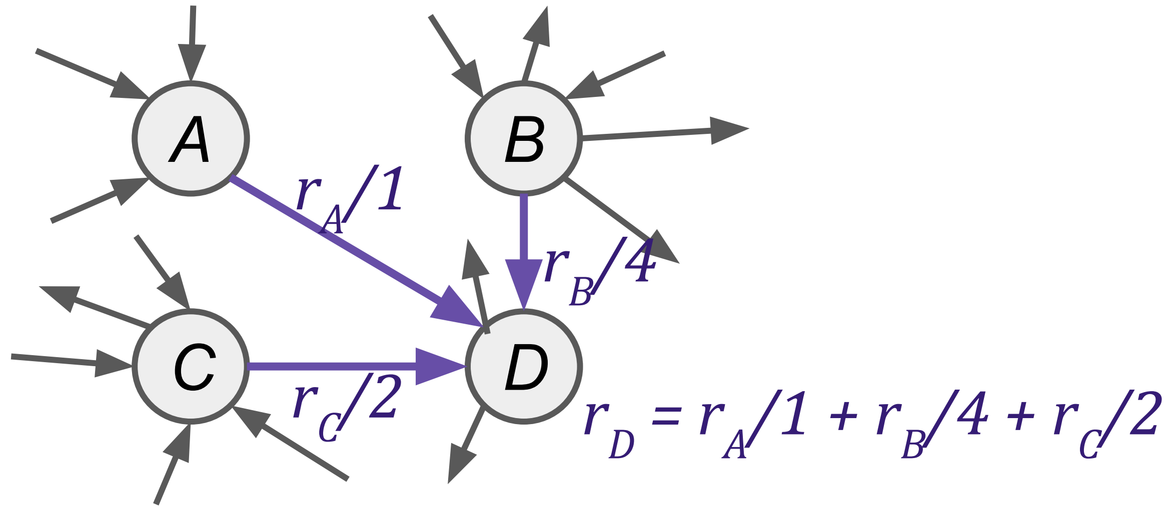

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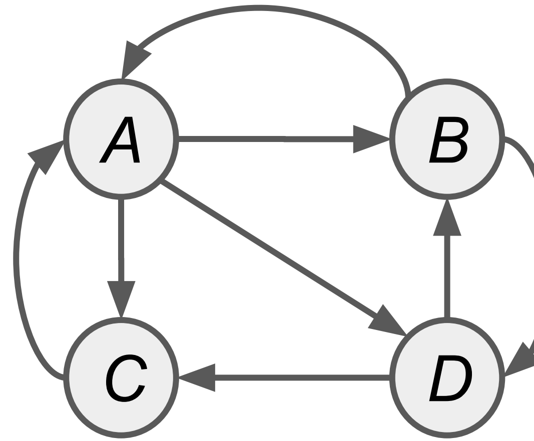

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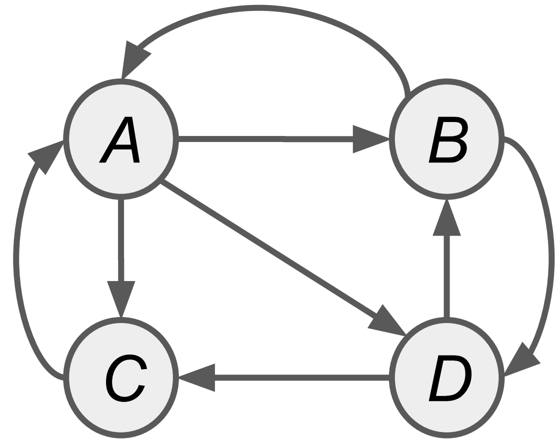

View 1: Flow Model:

A System of Equations:

$$r_A = \frac{r_B}{2} + \frac{r_C}{1}$$

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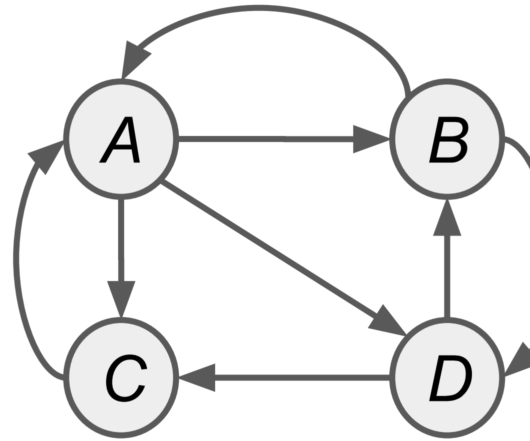

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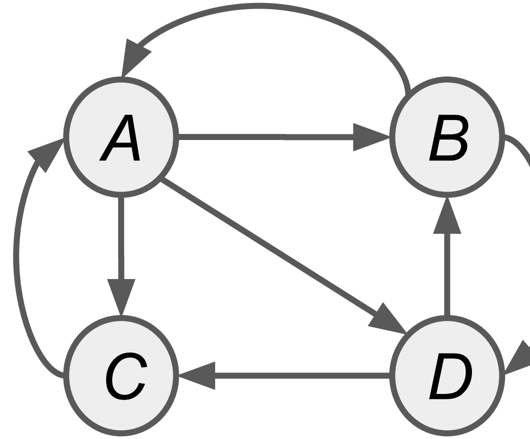
PageRank

View 1: Flow Model: Solve

$$1 = r_A + r_B + r_C + r_D$$



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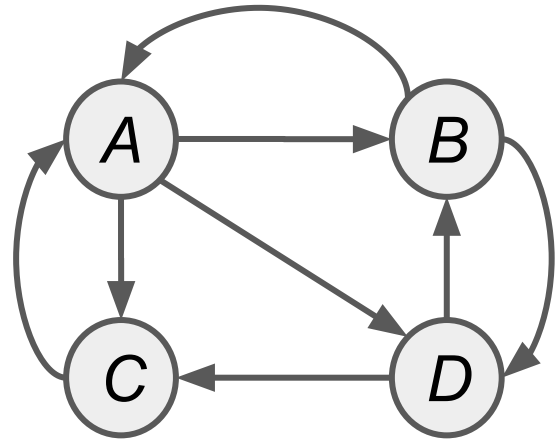

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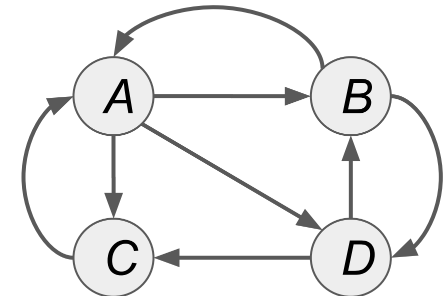
Transition Matrix, M

PageRank

View 2: Matrix Formulation



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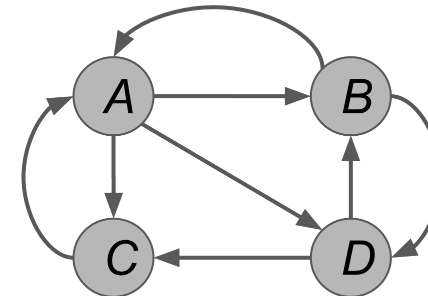
Transition Matrix, M

Innovation: What pages would a "random Web surfer" end up at?



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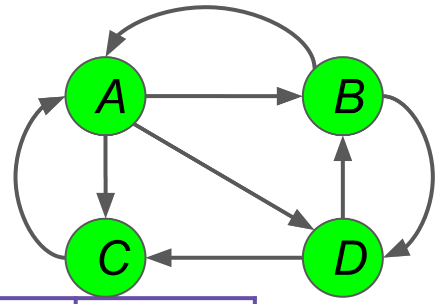
Transition Matrix, M

Innovation: What pages would a "random Web surfer" end up at?

```
To Start, all are equally likely at ¼
```



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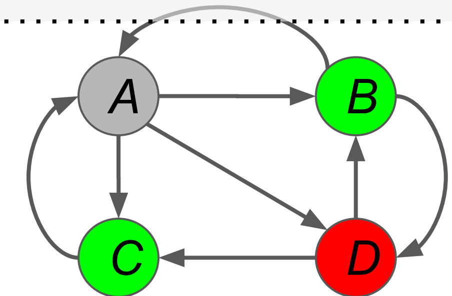
Transition Matrix, M

Innovation: What pages would a "random Web surfer" end up at?

To Start, all are equally likely at ¼: ends up at D



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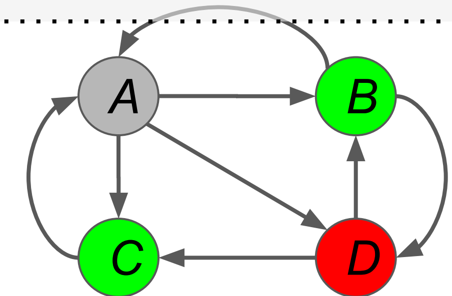
Transition Matrix, M

Innovation: What pages would a "random Web surfer" end up at?

To Start, all are equally likely at ¼: ends up at D

C and B are then equally likely: ->D->B=¼*½; ->D->C=¼*½



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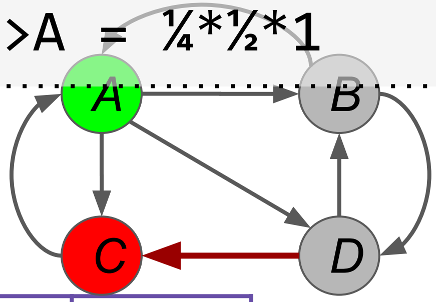
Ends up at C: then A is only option: ->D->C->A = ¼*½*1



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Transition Matrix, M

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. . .

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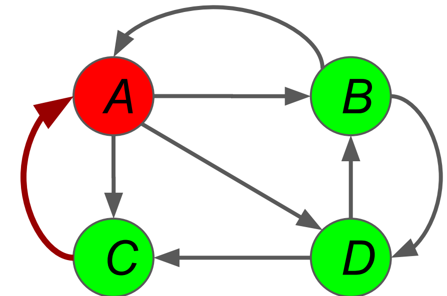

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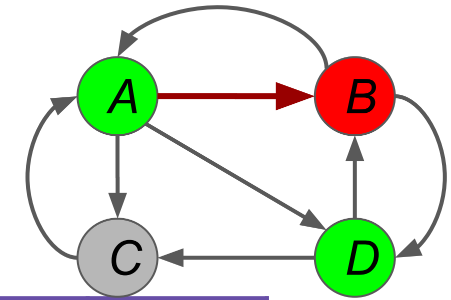

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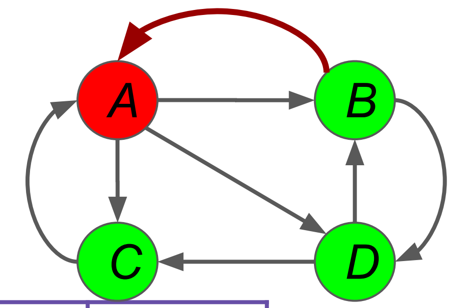

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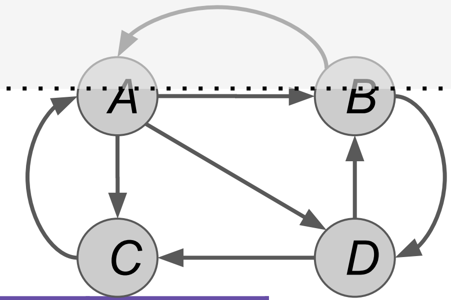
To start: N=4 nodes, so r = [¼, ¼, ¼, ¼,]



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Innovation: What pages would a “random Web surfer” end up at?

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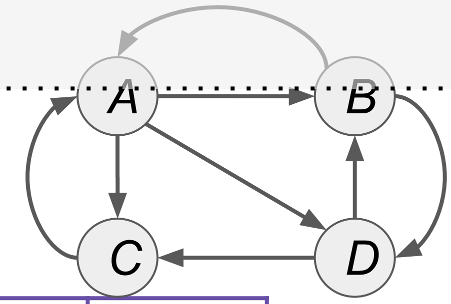
after 1st iteration: $M \cdot r$ = [3/8, 5/24, 5/24, 5/24]



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Transition Matrix, M

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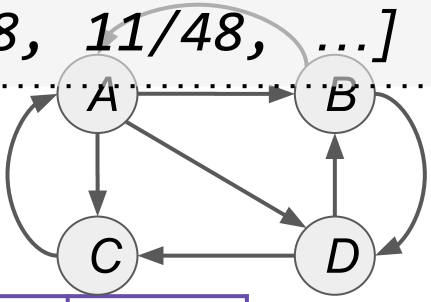
after 1st iteration: $M \cdot r$ = [3/8, 5/24, 5/24, 5/24]

after 2nd iteration: $M(M \cdot r) = M^2 \cdot r$ = [15/48, 11/48, ...]



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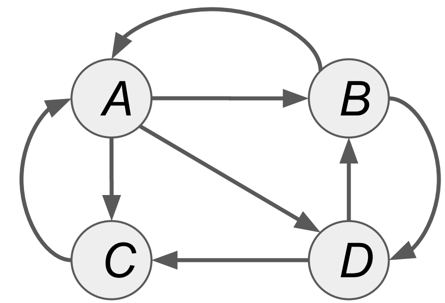
Power iteration algorithm

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
while (err_norm(r[t],r[t-1])>min_err):



err_norm(v1, v2) = |v1 - v2| #L1 norm
```



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"Transition Matrix", *M*

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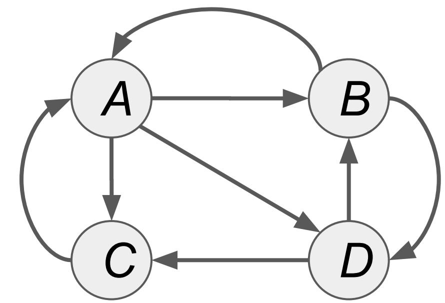

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“Transition Matrix”, *M*

As err_norm gets smaller we are moving toward: $r = M \cdot r$

View 3: Eigenvectors:

Power iteration algorithm

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View 3: Eigenvectors:

We are actually just finding the ***eigenvector*** of M.

finds the...

Power iteration algorithm

```
initialize:  r[0] = [1/N, …, 1/N]
             r[-1]=[0,...,0]
while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = M·r[t]
    t+=1
solution = r[t]

err_norm(v1, v2) = |v1 - v2| #L1 norm
```

x is an *eigenvector* of A if:
$A \cdot \mathbf{x} = \lambda \cdot \mathbf{x}$

(Leskovec at al., 2014; http://www.mmds.org/)

As err_norm gets smaller we are moving toward: $r = M \cdot r$

View 3: Eigenvectors:

We are actually just finding the ***eigenvector*** of M.

finds the...

Power iteration algorithm

```
initialize:  r[0] = [1/N, …, 1/N]
             r[-1]=[0,...,0]
while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = M·r[t]
    t+=1
solution = r[t]

err_norm(v1, v2) = sum(|v1 - v2|)
                          #L1 norm
```

x is an *eigenvector* of A if:
$A \cdot \mathbf{x} = \lambda \cdot \mathbf{x}$

$\lambda = 1$ (eigenvalue for 1st principal eigenvector)
since columns of M sum to 1.
Thus, *if* r is **x**, then $Mr = 1r$

View 4: Markov Process

Where is surfer at time t+1? $p(t+1) = M \cdot p(t)$

Suppose: $p(t+1) = p(t)$, then $p(t)$ is a *stationary distribution* of a ***random walk***.

Thus, r is a stationary distribution. Probability of being at given node.

View 4: Markov Process

Where is surfer at time t+1? $p(t+1) = M \cdot p(t)$

Suppose: p(t+1) = p(t), then p(t) is a *stationary distribution* of a ***random walk***.

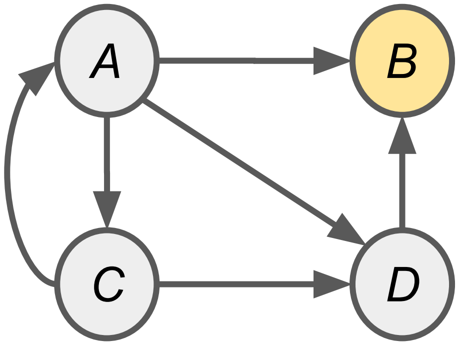
Thus, r is a stationary distribution. Probability of being at given node.

aka 1st order Markov Process

- Long history in probabilistic theory. One finding:
  - Stationary distributions have a unique distribution if:
    - No *"dead-ends"*: a node can't propagate its rank
    - No *"spider traps"*: set of nodes with no way out.

Also known as being *stochastic*, *irreducible*, and *aperiodic.*

View 4: Markov Process - Problems for vanilla PI



to \ from	***A***	***B***	***C***	***D***
A	0	0	1	0
B	1/3	0	0	1
C	1/3	0	0	0
D	1/3	0	0	0

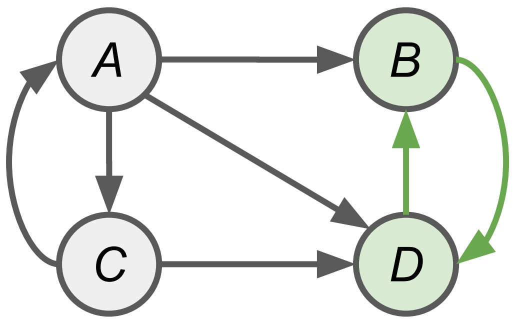
What would *r* converge to?

aka 1st order Markov Process

- Rich probabilistic theory. One finding:
  - Stationary distributions have a unique distribution if:
    - No "***dead-ends***": a node can't propagate its rank
    - No "*spider traps*": set of nodes with no way out.

Also known as being *stochastic*, *irreducible*, and *aperiodic.*

View 4: Markov Process - Problems for vanilla PI



to \ from	**A**	**B**	**C**	**D**
A	0	0	1	0
B	1/3	0	0	1
C	1/3	0	0	0
D	1/3	1	0	0

What would *r* converge to?

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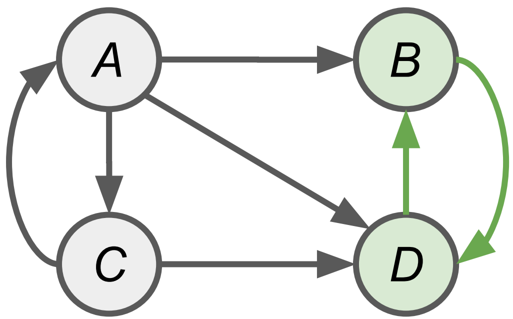
View 4: Markov Process - Problems for vanilla PI



to \ from	***A***	***B***	***C***	***D***
A	0	0	1	0
B	1/3	0	0	1
C	1/3	0	0	0
D	1/3	1	0	0

What would *r* converge to?

aka 1st order Markov Process

- Rich probabilistic theory. One finding:
  - Stationary distributions have a unique distribution if:

same node doesn't repeat at regular intervals

columns sum to 1 non-zero chance of going to any other node

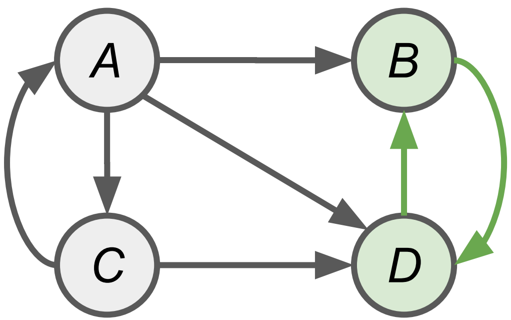
Also known as being *stochastic*, *irreducible*, and *aperiodic*.

Goals:
No “dead-ends”
No “spider traps”

The “Google” PageRank Formulation

Add teleportation:At each step, two choices

1. Follow a random link (probability, β = ~.85)
2. Teleport to a random node (probability, 1-β)



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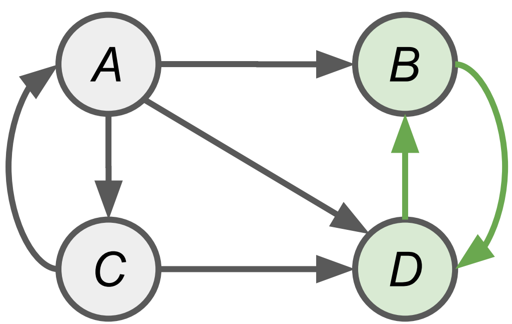

to \ from	***A***	***B***	***C***	***D***
A	0	0	1	0
B	⅓	0	0	1
C	⅓	0	0	0
D	⅓	1	0	0

Goals:
No "dead-ends"
No "spider traps"

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1. Follow a random link (probability, β = ~.85)
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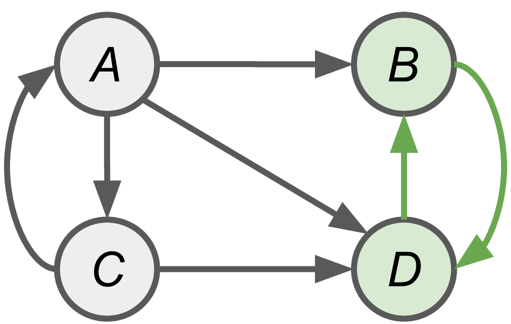

to \ from	***A***	***B***	***C***	***D***
A	0	0+.15*¼	1	0+.15*¼
B	⅓	0+.15*¼	0	.85*1+.15*¼
C	⅓	0+.15*¼	0	0+.15*¼
D	⅓	.85*1 +.15*¼	0	0+.15*¼

Goals:
No "dead-ends"
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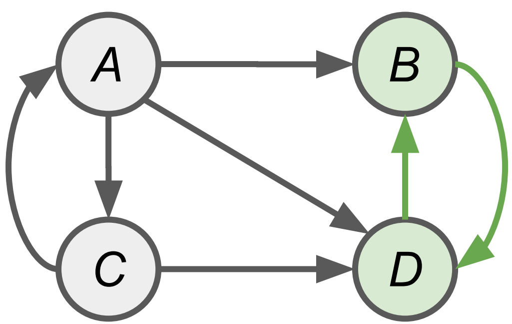

to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	0+.15*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	0+.15*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	0+.15*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	.85*1+.15*¼	0+.15*¼	0+.15*¼

Goals:
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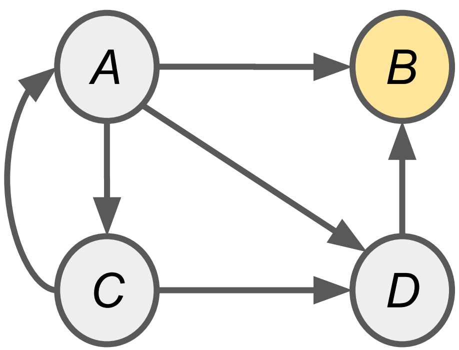

to \ from	***A***	***B***	***C***	***D***
A	0	0	1	0
B	⅓	0	0	1
C	⅓	0	0	0
D	⅓	0	0	0

Goals:
No “dead-ends”
No “spider traps”

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Add teleportation:At each step, two choices

1. Follow a random link (probability, β = ~.85)
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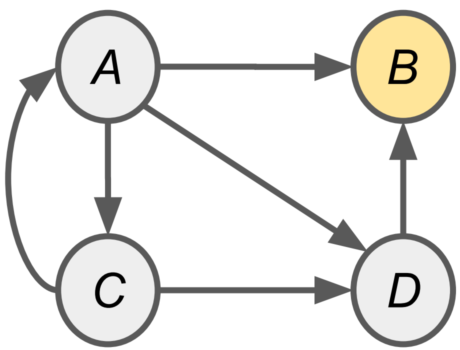


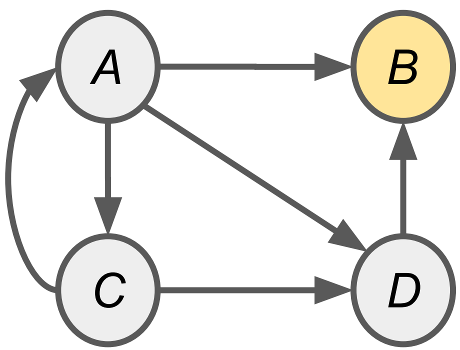
to \ from	***A***	***B***	***C***	***D***
A	0	¼	1	0
B	⅓	¼	0	1
C	⅓	¼	0	0
D	⅓	¼	0	0

Goals:
No "dead-ends"
No "spider traps"

The "Google" PageRank Formulation

Add teleportation:At each step, two choices

1. Follow a random link (probability, β = ~.85)
2. Teleport to a random node (probability, 1-β)



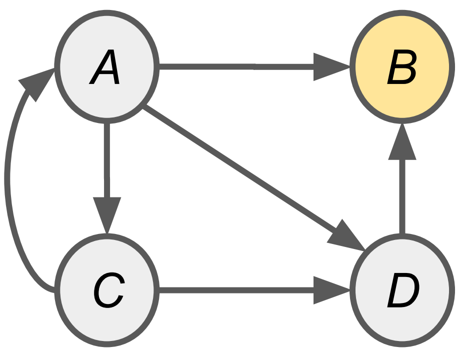
to \ from	***A***	***B***	***C***	***D***
A	0	.85*¼+.15*¼	1	0
B	⅓	.85*¼+.15*¼	0	1
C	⅓	.85*¼+.15*¼	0	0
D	⅓	.85*¼+.15*¼	0	0

Goals:
No “dead-ends”
No “spider traps”

The “Google” PageRank Formulation

Add teleportation:At each step, two choices

1. Follow a random link (probability, β = ~.85)
2. Teleport to a random node (probability, 1-β)
   (Teleport from a dead-end has probability 1)



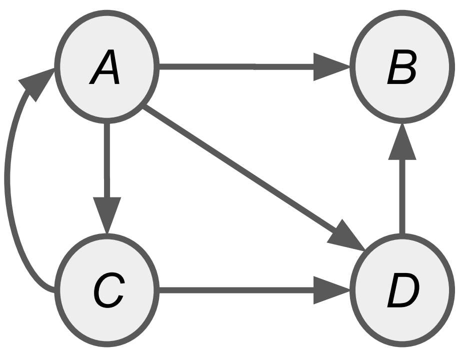
to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	1*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	1*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

Goals:
No “dead-ends”
No “spider traps”

Teleportation, as Flow Model:

$$r_j = \sum_{i \rightarrow j} \beta \, \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

(Brin and Page, 1998)



to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	1*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	1*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

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$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

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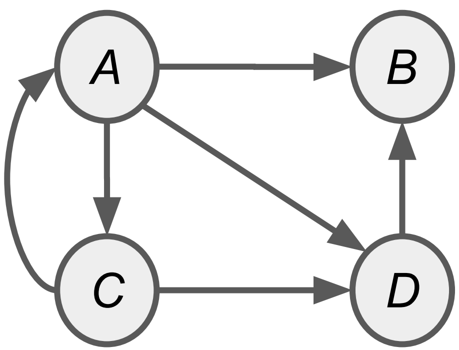
Teleportation, as Matrix Model:

$$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N}$$



to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	1*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	1*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

Goals:
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$$r_j = \sum_{i \to j} \beta \, \frac{r_i}{d_i} + (1 - \beta) \frac{1}{N}$$

(Brin and Page, 1998)

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$$M' = \beta \, M + (1 - \beta) \left[\frac{1}{N} \right]_{N \times N}$$

to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	.85*¼+.15*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	.85*¼+.15*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	.85*¼+.15*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	.85*¼+.15*¼	0+.15*¼	0+.15*¼

Goals:
No "dead-ends"
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$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1 - \beta) \frac{1}{N}$$

(Brin and Page, 1998)

Teleportation, as Matrix Model:

$$M' = \beta M + (1 - \beta) \left[\frac{1}{N}\right]_{N \times N}$$

To apply:
run power iterations over M' instead of M.

to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	1*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	1*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

Goals:
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Teleportation, as Flow Model:

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

(Brin and Page, 1998)

Teleportation, as Matrix Model:

$$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N}$$

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
3. Convert *M* to *M’*
4. Run Power Iterations.

to \ from	***A***	***B***	***C***	***D***
A	0+.15*¼	1*¼	85*1+.15*¼	0+.15*¼
B	.85*⅓+.15*¼	1*¼	0+.15*¼	.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

Goals:
No "dead-ends"
No "spider traps"

Teleportation, as Flow Model:

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{N}$$

(Brin and Page, 1998)

Teleportation, as Matrix Model: $M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N}$

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
3. Convert *M* to *M'*
4. Run Power Iterations.

But, M' is now a dense matrix!

E.g. 50B webpages as nodes.
50B x 50B = 2.5 x 10^{21}!

to	[illegible]	[illegible]	[illegible]	D
[illegible]	[illegible]	[illegible]	[illegible]	0+.15*¼
B	[illegible]	[illegible]	[illegible]	.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

PageRank, large scale

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
3. Convert *M* to *M'*
4. Run Power Iterations.

Teleportation, as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

But, M' is now a <u>dense</u> matrix!

E.g. 50B webpages as nodes.
50B x 50B = 2.5 x 10^{21}!

to				**D**
				0+.15*¼
				.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

PageRank, large scale

... M is sparse (mostly 0s)...

Teleportation, as Matrix Model: $M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N}$

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
3. Convert *M* to *M'*
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But, M' is now a <u>dense</u> matrix!

E.g. 50B webpages as nodes.
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to				*D*
				0+.15*¼
				.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

PageRank, large scale

… M is sparse… Can we just work with M?

Teleportation, as Matrix Model: $M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
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But, M' is now a dense matrix!

E.g. 50B webpages as nodes.
50B x 50B = 2.5 x 10^{21}!

to				***D***
				0+.15*¼
				.85*1+.15*¼
C	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼
D	.85*⅓+.15*¼	1*¼	0+.15*¼	0+.15*¼

PageRank, large scale

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Teleportation, as Matrix Model: $M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
3. Convert *M* to *M'*
4. Run Power Iterations.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]


while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = M·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

... M is sparse... Can we just work with M?

Store as a sparse representation

$$... M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

Steps:

1. **Compute M**
2. Add 1/N to all dead-ends.
3. Convert *M* to *M'*
4. Run Power Iterations.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]


while (err_norm(r[t],r[t-1])>min_err):
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```

PageRank, large scale

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Teleportation, as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

Steps:

1. Compute M
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3. Convert *M* to *M'*
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```

PageRank, large scale

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Steps:

1. Compute M
2. **Add 1/N to all dead-ends.**
3. Convert *M* to *M'*
4. Run Power Iterations.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)

while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = Mnod·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

... M is sparse... Can we just work with M?

Teleportation, as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N}$$

Steps:

1. Compute M
2. Add 1/N to all dead-ends.
3. **Convert *M* to *M'***
4. Run Power Iterations.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
M’ = beta*Mnod + (1-beta)*[1/N]NxN
while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = M’·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

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Teleportation, as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

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    r[t+1] = M’·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

… M is sparse… *Can we just work with M?*

Teleportation, as Matrix Model:

$$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

Yes! Work with the calculation of M' within the inner loop.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
M’ = beta*Mnod + (1-beta)*[1/N]NxN
while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = M’·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

... M is sparse... Can we just work with M?

Teleportation, as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

Yes! Work with the calculation of M' within the inner loop.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
M’ = ~~beta*Mnod + (1-beta)*[1/N]NxN~~
while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = (beta*Mnod + (1-beta)*[1/N]NxN)·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

... M is sparse... Can we just work with M?

Teleportation,
as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$$

```
initialize:  r[0] = [1/N, …, 1/N],
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Mnod = addToDeadEnds(1/N, M)

while (err_norm(r[t],r[t-1])>min_err):
    r[t+1] = (beta*Mnod + (1-beta)*[1/N]_NxN)·r[t]
    t+=1
solution = r[t]
```

PageRank, large scale

... M is sparse... Can we just work with M?

Teleportation, as Matrix Model: $$M' = \beta\, M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N}$$

The second half of the M' equation is just a constant

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
tele = (1-beta)* (1/N)
While (err_norm(r[t],r[t-1])>min_err):
  r[t+1] = (beta*Mnod + (1-beta)*[1/N]NxN)·r[t]
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tele = (1-beta)* (1/N)
while (err_norm(r[t],r[t-1])>min_err):
  r[t+1] = (beta*Mnod + tele)·r[t]
  t+=1
solution = r[t]
```

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Anything else we can do to save space or computation?

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
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while (err_norm(r[t],r[t-1])>min_err):
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```

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Is M larger than it needs to be because of the dead-ends?

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
tele = (1-beta)* (1/N)
while (err_norm(r[t],r[t-1])>min_err):
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Exercise:

Get rid of this step. How to adjust algorithm?

Hint: at least 2 options:

1. Track dead ends

2. Consider that:

r should sum to 1.

```
initialize:  r[0] = [1/N, …, 1/N],
             r[-1]=[0,...,0]
Mnod = addToDeadEnds(1/N, M)
tele = (1-beta)* (1/N)
while (err_norm(r[t],r[t-1])>min_err):
   r[t+1] = (beta*Mnod + tele)·r[t]
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solution = r[t]
```

PageRank: Summary

- Flow View: Link Voting
- Matrix View: Linear Algebra
  - Eigenvectors View
- Markov Process View
- How to remove:
  - Dead Ends
  - Spider Traps

In practice, sparse matrix, implement teleportation functionally rather than update M’

Search, 20+ years later

The Anatomy of a Large-Scale Hypertextual Web Search Engine

Sergey Brin and Lawrence Page

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Stanford University, Stanford, CA 94305, USA
sergey@cs.stanford.edu and page@cs.stanford.edu

Abstract

In this paper, we present Google, a prototype of a large-scale search engine which makes heavy use of the structure present in hypertext. Google is designed to crawl and index the Web efficiently and produce much
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Many innovations since
examples:

- Content Specific, "Personalized PageRank"
- Search Engine Optimization (SEO) countermeasures
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but still core of approach: PageRank